Indian Journal of Modern Research and Reviews

This Journal is a member of the *'Committee on Publication Ethics'*

Online ISSN:2584-184X

Research Article

A Study on Iterated Line Graphs

Athira P. V. [1*], Dr. Ambat Vijayakumar [2]

[1] Assistant Professor, Department of Mathematics, Mahatma Gandhi University Priyadarsini Hills, Athirampuzha, Kottayam — 686560, Kerala, India

[2] Emeritus Professor, Department of Mathematics, Cochin University of Science and Technology, Kochi, Kerala, India

Corresponding Author: * Athira P. V.

DOI: https://doi.org/10.5281/zenodo.20476767

Abstract

The line graph *L(G)* of an undirected graph *G* represents the adjacencies between the edges of *G*. Iterated line graphs are defined recursively as *Lk(G) = L(Lk-1(G))* for $k \geq 1$. This dissertation investigates the behavior and evolution of classical structural properties and parameter values—such as order, size, maximum/minimum degree, chromatic number, clique number, and vertex/edge connectivity—under repeated line-graph operators. We characterize the limits of these behaviors based on the initial structural configurations of prolific graphs. Furthermore, we explore topological chemical indices within this operational framework, focus extensively on structural behaviors minimizing or preserving the Wiener index across iterations, and establish exact bounds and configurations for families of trees, caterpillars, lobsters, and generalized stars.

Manuscript Information

- **ISSN No:** 2584-184X
- **Received:** 13-04-2026
- **Accepted:** 23-05-2026
- **Published:** 31-05-2026
- **MRR:**4(5); 2026: 366-370
- **©2026, All Rights Reserved**
- **Plagiarism Checked:** Yes
- **Peer Review Process:** Yes

How to Cite this Article

Athira P. V., Dr. Ambat Vijayakumar. A Study on Iterated Line Graphs. Indian J Mod Res Rev. 2026;4(5): 366-370.

Access this Article Online

www.mrrjournal.in

KEYWORDS: Vertex Set, Wiener Index, Leonhard Euler, Jensen's Inequality, Hamiltonian Dynamics

INTRODUCTION

In mathematics, graph theory is the study of graphs, which are mathematical structures used to model pairwise relations between objects. A graph in this context is made up of vertices (also called nodes or points) which are connected by edges (also called arcs, links, or lines). The basic framework of graphs was introduced in the eighteenth century by Leonhard Euler in 1736 to resolve the Konigsberg bridge problem. While graph theory was initially viewed as an isolated branch of recreational mathematics, recent developments have provided a strong impetus for its rigorous expansion across areas like network architectures, chemistry, and electrical circuits.

A graph $G = (V(G), E(G))$ consists of a non-empty set of vertices $V(G)$ and a set of edges $E(G)$. Each edge $e \in E$ has an associated pair of vertices called its endpoints. The line graph of an undirected graph G is the graph $L(G)$ that represents the adjacencies between the edges of G. Formally, $L(G)$ is constructed by mapping each edge of G to a unique vertex in $L(G)$, where two vertices in $L(G)$ are connected by an edge if and only if their corresponding edges in G share a common vertex. The two foundational characterizations of line graphs are established by Krausz and Beineke.

Given a graph G, we denote the kth iterated line graph of G by $L^k(G)$, where

$$L^k(G) = L(L^{k-1}(G)),\ k \geq 1,$$

with

$$L^0(G) = G.$$

In this work, we study the numerical and qualitative behavior of classic parameters under iteration. We focus specifically on prolific graphs: connected graphs that are not isomorphic to any path P_k, any cycle C_k, or the claw graph $K_{1,3}$.

Following the seminal work of Van Rooij and Wilf (1965), the sequence of iterated line graphs for a finite connected graph exhibits exactly four behaviors:

- If G is a cycle graph, then $L(G)$ and each subsequent iteration are isomorphic to G.
- If G is the claw $K_{1,3}$, then $L(G)$ and all subsequent graphs are isomorphic to triangles (K_3).
- If G is a path, each subsequent iteration is a shorter path until the sequence terminates into an empty graph.
- In all remaining cases (prolific graphs), the size and order of the graphs increase without bound.

1. BASIC CONCEPTS & PREPARATORY TOOLS

Definition 1.1.
A graph is an ordered triple $G = (V(G), E(G), I_G)$, where $V(G)$ is a non-empty set of vertices, $E(G)$ is a set of edges disjoint from $V(G)$, and I_G is an incidence map associating each edge with an unordered pair of vertices.

Definition 1.2.
Let G be a loopless graph. The vertex set of the line graph $L(G)$ is in a one-to-one correspondence with the edge set of G, and two vertices of $L(G)$ are adjacent if and only if their corresponding edges share a common endpoint in G.

Definition 1.3.
A Hamiltonian cycle in a graph G is a cycle that visits every vertex of G exactly once and returns to the starting vertex. A graph is Eulerian if it contains a closed trail that traverses every edge exactly once.

Definition 1.4.
A star graph $K_{1,n-1}$ is a complete bipartite graph where $n-1$ vertices have degree 1 and a single central vertex has degree $n-1$. A caterpillar is a tree in which all vertices are within distance 1 of a central path, and a lobster is a tree that reduces to a caterpillar after removing all its pendant vertices.

Theorem 1.5 (Jensen's Inequality).
If f is a real continuous function that is convex on a given domain, then

$$f\left[(1/n) \sum_{i=1}^{n} x_i\right] \leq (1/n) \sum_{i=1}^{n} f(x_i)$$

Equality holds if and only if $x_1 = x_2 = \cdots = x_n$ or if f is strictly linear over the domain.

Theorem 1.6.
For any graph G with average degree $d(G)$, the average degree of its line graph $d_1(G)$ satisfies

$$d_1(G) \geq 2(d(G) - 1),$$

with equality holding if and only if G is regular.

Proof.
Let
$$n_1(G) = e(G) = \tfrac{1}{2} \sum \deg(v_j)$$

and
$$2e(G) = n \cdot d(G).$$

The number of edges in the line graph is given by

$$e_1(G) = \sum_{v \in V(G)} \binom{\deg(v)}{2}.$$

By applying Jensen's inequality to the convex function $f(x) = x^2$, we obtain

$$2e_1(G) = \sum \deg(v_j)^2 - 2e(G) \geq n \cdot d(G)^2 - n \cdot d(G).$$

Therefore,
$$d_1(G) = 2e_1(G)/n_1(G) \geq 2(d(G) - 1).$$

Theorem 1.7.
Let G be a connected graph and H a non-empty subgraph of G. Then

$e1(G) - e1(H) \geq e(G) - e(H)$.

2. BASIC PROPERTIES AND HAMILTONIAN DYNAMICS OF LINE GRAPHS

Line graphs translate structural characteristics from edge adjacencies to vertex relationships. The basic structural mappings are formalized as follows:

- A graph G is connected if and only if L(G) is connected.
- If $H \leq G$, then $L(H) \leq L(G)$.
- If G is r-regular, then L(G) is $(2r - 2)$-regular.
- If $e = uv \in E(G)$, its degree as a vertex in L(G) is given by $dL(G)(e) = dG(u) + dG(v) - 2$.

Summing across the vertex degrees of L(G) provides the exact edge count for the line graph system:

$m(L(G)) = ½ \Sigma(i=1 \text{ to } n)\ di^2 - m$

where (d1, d2, ..., dn) represents the degree sequence of G and m = e(G).

Theorem 2.1.
The line graph of a simple graph G is a path if and only if G is a path.

Proof.
If G is a path Pn, then $L(G) \cong Pn-1$. Conversely, suppose L(G) is a path. No vertex in G can have a degree greater than 2, because a vertex of degree ≥ 3 would produce a complete subgraph K3 in L(G), contradicting its path topology. Thus, G must be a path or a cycle; since $L(Cn) \cong Cn$, G must be a path.

Theorem 2.2 (Harary–Nash-Williams).
The line graph L(G) of a graph with at least three edges is Hamiltonian if and only if G contains a dominating trail, which is a closed trail such that every edge of G not belonging to the trail is incident to it.

Corollary 2.3.
If G is a connected graph with minimum degree $\delta(G) \geq 3$, then $L^2(G)$ is Hamiltonian.

Proof.
Since $\delta(G) \geq 3$, every vertex in L(G) is a member of a clique of size at least 3. Thus, every edge of L(G) is a component of a triangle, satisfying the requirements for L(G) to possess a dominating trail. Hence,

$L(L(G)) = L^2(G)$

is Hamiltonian.

3. INDEX EVOLUTION AND PARAMETER UNBOUNDEDNESS

For any structural parameter P(G), we define the parameter index for a graph family $\mathcal{F}$ as the minimum iterations required to guarantee a strict monotonic increase in the parameter's value.

Definition 3.1.
The parameter index of a prolific graph G is denoted by
$ind(P,G) = \min\{r : P(G) < P(Lr(G))\}$.

For a family $\mathcal{F}$ of prolific graphs, the family index is
$k(P,\mathcal{F}) = \max\{ind(P,G) : G \in \mathcal{F}\}$.

3.1 Order and Size Indices (n(G) and e(G))

Theorem 3.2.
Let G be a prolific graph on $n \geq 4$ vertices. Then

$k(n,\mathcal{F}) = 4$

and

$k(e,\mathcal{F}) = 2$.

Proof.
If $e(G) \geq n + 1$, then
$n_1(G) = e(G) > n$,

meaning ind(n,G) = 1. If e(G) = n (unicyclic containing a pendant path), then $n_2(G) > n$. When $e(G) = n - 1$, G is a tree. Analyzing structural tree subdivisions shows that if $x_3 = 1$ and G is a Type A subdivision of K1,3, the vertex count remains constant until the fourth iteration, where

$n_4(G) \geq n + 3$.

This establishes the tight upper bound
$k(n,\mathcal{F}) = 4$.

3.2 Extreme Degrees ($\Delta(G)$ and $\delta(G)$)

Definition 3.3.
A connected graph G is defined as fine if it contains at least one edge e = uv such that

$\deg(u) + \deg(v) - 2 > \Delta(G)$.

Theorem 3.4.
Let G be a prolific graph with maximum degree $\Delta \geq 3$. Then $k(\Delta,\mathcal{F}) = 3$.

Theorem 3.5.
Let $\mathcal{F}$ be the family of all prolific graphs. If
$\mathcal{F}_1 = \{G : \delta(G) \in \{1,2\}\}$,
then

$k(\delta,\mathcal{F}_1) = \infty$.

However, if

$\mathcal{F}_2 = \{G : \delta(G) \geq 3\}$,
then
$k(\delta,\mathcal{F}_2) = 1$.

Proof.
For graphs containing a long path segment of length m ending in a leaf, the vertex of degree 1 requires m line-graph iterations to be eliminated, pushing the index toward infinity as path length grows.

For $\delta \geq 3$,
$\delta_1(G) \geq 2\delta(G) - 2 > \delta(G)$

occurs within a single step.

3.3 Chromatic Number and Clique Number ($\chi(G)$ and $\omega(G)$)

Theorem 3.6.
For the family of all prolific graphs,

$k(\chi,\mathcal{F}) = 3$

and

$k(\omega,\mathcal{F}) = 3$.

Proof.
By Vizing's theorem,
$\chi_1(G) = \chi'(G) \in \{\Delta(G), \Delta(G)+1\}$.
Analyzing the configuration CP(3,n−3) (a triangle with an attached path) shows that
$\chi(G) = \chi_1(G) = \chi_2(G) = 3$,

and a strict increase is achieved at the third step with
$\chi_3(G) = 4$.

This proves that
$k(\chi,\mathcal{F}) = 3$.

For the clique index,
$ind(\omega,G) = 3$
occurs exclusively if $\omega = 4$ for G = K4 or when $\omega = \Delta = 3$ with the vertices of maximum degree forming an independent set.

4. WIENER INDEX OPTIMIZATION IN ITERATED LINE SYSTEMS

The Wiener index W(G) is a classical chemical descriptor defined as the sum of distances between all unordered pairs of vertices:

$W(G) = \Sigma\{u,v \subseteq V(G)\}\ d(u,v)$

Theorem 4.1.
If G is a connected unicyclic graph of order n, then
$W(L(G)) \leq W(G)$,

with equality holding if and only if

$G \cong Cn$.

Theorem 4.2.
Let T be a tree of order $n \geq 2$. Then
$W(L(T)) = W(T) - C(n,2)$

where
$C(n,2) = n(n-1)/2$.
Consequently, the Wiener indices of a tree and its first line graph are always distinct.

This structural property raises an intriguing mathematical question: do there exist configurations satisfying
$W(L^2(T)) = W(T)$?
Comprehensive computational searches across all trees of order $n \leq 26$ reveal exactly how these trees are distributed, as summarized in Table 4.1.

Table 4.1: Distribution of Trees of Order n Satisfying W(L2(T)) = W(T)

Order (*n*)	Total Trees (*tn*)	Invariant Trees (*wn*)	Order (*n*)	Total Trees (*tn*)	Invariant Trees (*wn*)
9	47	1	18	123,867	73
10	106	1	19	317,955	204
11	235	1	20	823,065	231
12	551	0	21	2,144,505	513
13	1,301	7	22	5,623,756	576
14	3,159	8	23	14,828,074	1,520
15	7,741	22	24	39,299,897	1,715
16	19,320	25	25	104,636,890	3,763
17	48,629	66	26	279,793,450	4,085

Theorem 4.3.
There exist infinite structural families of trees—specifically within the classes of lobsters and caterpillars—that satisfy the second-order line-graph invariance property

$W(L^2(T)) = W(T)$.

Proof.
We construct an explicit infinite family of lobsters Tk characterized by central branches of precisely defined lengths:
$xk = (3/2)(k^2 - k + 2)$

and
$y_k = (3/2)(k^2 + k + 2), \quad k \geq 0.$

Because
$x_{k+1} = y_k,$

consecutive configurations match along asymmetric dimensions, allowing the net distance gains in $L^2(T)$ to perfectly balance out the structural losses from $L(T)$. This algebraic balancing confirms the existence of an infinite family.

Theorem 4.4.
Let S be a generalized star featuring Δ branches of lengths $k_1, k_2, ..., k_\Delta$ around a central vertex, and let $q = e(S)$. If $\Delta = 3$, then

$W(L^2(S)) < W(S).$

If $\Delta \geq 7$, then

$W(L^2(S)) > W(S).$

Consequently, any generalized star with invariant second-order behavior must have a maximum degree

$\Delta \in \{4, 5, 6\}.$

CONCLUSION

This study demonstrates how structural features evolve under iterated line-graph operators. We established tight upper bounds for the indices of classical structural parameters—such as order, size, maximum and minimum degree, chromatic number, and clique number—over the family of prolific graphs. We also provided a structural characterization of graphs that preserve the Wiener index across iterations.

These structural findings show that balancing local edge densities against global path expansions can keep graph parameters stable under iteration. This operational framework opens up several paths for future research, particularly regarding the bounds of independence numbers, domination constraints, and spectral gap expansions in iterated graph systems.

REFERENCES

1. Beineke LW. Characterizations of derived graphs. J Combin Theory. 1970;9(2):129-135.
2. Chartrand G. On hamiltonian line-graphs. Trans Am Math Soc. 1968;134(3):559-566.
3. Krausz J. Démonstration nouvelle d'un théorème de Whitney sur les réseaux. Mat Fiz Lapok. 1943;50(1):75-85.
4. van Rooij ACM, Wilf HS. The interchange graph of a finite graph. Acta Math Acad Sci Hung. 1965;16(3):263-269.
5. Chartrand G, Gavlas H, Schultz M. Convergent sequences of iterated H-line graphs. Discrete Math. 1995;147:73-86.
6. Sumner DP. Graphs with 1-factors. Proc Am Math Soc. 1974;42(1):8-12.
7. Stacho L. New upper bounds for the chromatic number of a graph. J Graph Theory. 2001;36(2):117-120.
8. Vizing VG. On an estimate of the chromatic class of a p-graph. Discret Analiz. 1964;3:25-30.
9. Whitney H. A theorem on graphs. Ann Math. 1931;32(2):378-390.
10. Caro Y, Lauri J, Zarb C. Index of parameters of iterated line graphs. Discrete Math. 2022;345(8):112932.
11. Knor M, Potocnik P, Skrekovski R. The Wiener index in iterated line graphs. Discrete Appl Math. 2020;160:2234-2245.
12. Dobrynin AA, Mel'nikov LS. Some results on the Wiener index of iterated line graphs. Electron Notes Discrete Math. 2005;22:469-475.

About the Corresponding Author

Athira P. V. is an Assistant Professor in the Department of Mathematics at Mahatma Gandhi University, Priyadarsini Hills, Athirampuzha, Kottayam, Kerala, India. She is engaged in teaching and research in mathematics, contributing to the advancement of mathematical sciences through academic scholarship and student mentoring. Her research interests include contemporary areas of pure and applied mathematics, and she actively participates in academic and research activities within the mathematical community.